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Behavior of Solutions of Certain Second-Order Differential Equations at Infinity

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This article is concerned with solutions of

$$u'' + fu = 0. \qquad t \geqslant 0. \tag{1}$$

$t \to \infty$ and all hypotheses hold for $t \geqslant T$ where $T = T(u, f) \gg 1$. It is assumed that

$$\liminf_{t \to \infty} t^2 f(t) > \tfrac{1}{4}, \tag{2}$$

so all solutions are oscillatory. We define

$$g = \frac{(f^{-1/4})''}{f^{-1/4}} = \frac{5}{16}\left(\frac{f'}{f}\right)^2 - \frac{1}{4}\left(\frac{f''}{f}\right). \tag{3}$$

The letters a, b denote constants, with $a \geqslant 1$ and $b > 0$. A statement such as $g \leqslant o(f)$ means $g \leqslant \varepsilon f$ where $\varepsilon \to 0$ as $t \to \infty$.

A remarkable theorem of Wintner [1, p. 371] asserts the following: Suppose

$$\int^{\infty} f(t)^{1/2}\, dt = \infty \qquad \text{and} \qquad \int^{\infty} \frac{|g(t)|}{f(t)^{1/2}}\, dt < \infty. \tag{4}$$

Then there are constants A, B such that

$$f^{1/4} u = (A + o(1)) \cos \int_0^t \sqrt{f(s)}\, ds + (B + o(1)) \sin \int_0^t \sqrt{f(s)}\, ds. \tag{5}$$

In paticular, $u = O(f^{-1/4})$.

This article arose in connection with a new proof of the latter estimate, which was needed for certain special cases in [2]. Our final conclusion

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$u = O(f^{-1/4})$ is weaker than Wintner's conclusion (5) and our final hypothesis is stronger than his. But on the way we establish several lemmas and results of independent interest, and hypotheses for these do not imply (4). The methods are quite different from those of Wintner as presented in [1] and may prove useful in other applications.

Let u satisfy (2) and define

$$v(t) = f(t)^{-1/4} \sin aF(t), \qquad F(t) = \int_\alpha^t \sqrt{f(s)}\, ds.$$

Then by a short calculation $v'' + (a^2 f - g) v = 0$, which gives:

LEMMA 1. *Let α and β be two consecutive zeros of u and suppose $g \geqslant 0$. Then*

$$0 \leqslant \frac{u(t)}{u'(\alpha)} \leqslant \frac{1}{f(t)^{1/4} f(\alpha)^{1/4}} \sin F(t) \qquad \alpha \leqslant t \leqslant \beta.$$

For proof, set $w(t) = u'(\alpha)\, u(t)/(af(\alpha)^{1/4})$ and note that $w(\alpha) = u(\alpha)$, $w'(\alpha) = u'(\alpha)$. The conclusion follows from Sturm's theorem applied to the case $a = 1$.

LEMMA 2. *In Lemma* 1 *suppose $0 \leqslant g \leqslant (a^2 - 1) f$. Then*

$$\pi \geqslant \int_\alpha^\beta f(s)^{1/2}\, ds \geqslant \frac{\pi}{a}.$$

The first inequality follows from Lemma 1. For the second, the hypothesis $a^2 f - g \geqslant f$ and Sturm's comparison theorem give $0 \leqslant w(t) \leqslant u(t)$ for $\alpha \leqslant t \leqslant \gamma$, where γ is defined by

$$a \int_\alpha^\gamma f(s)^{1/2}\, ds = \pi.$$

From $u(\beta) = 0$ we get $\gamma \leqslant \beta$ and this yields Lemma 2.

Taking $a = a_n \to 1$ in Lemma 2, we get:

THEOREM 1. *Let $\{\alpha_n\}$ be zeros of u arranged in increasing order, and suppose $0 \leqslant g \leqslant o(f)$. Then*

$$\int_{\alpha_n}^{\alpha_{n+1}} f(t)^{1/2}\, dt \leqslant \pi \qquad \textit{and} \qquad \lim_{n \to \infty} \int_{\alpha_n}^{\alpha_{n+1}} f(t)^{1/2}\, dt = \pi.$$

When f is strictly increasing and $f(\infty)=\infty$ we define a sequence $\{t_j\}$ by

$$f(t_j)=j^2, \qquad j\gg 1.$$

The interval (t_j, t_{j+1}) is denoted by I_j and its length by I_j.

THEOREM 2. *With f and I_j as above suppose $g\leqslant(a^2-1)f$ and $f'\geqslant bf$. Let N_j denote the number of zeros of u in I_j. Then $N_j\leqslant 1+a/b$, $j\geqslant 3$.*

For a proof, the mean-value theorem for derivatives yields

$$f(t_{j+1})-f(t_j)=(t_{j+1}-t_j)\,f'(\xi)\geqslant(t_{j+1}-t_j)\,bj^2$$

where $\xi=\xi_j\in I_j$. The left side is $2j+1$, so $|I_j|\leqslant(2j+1)/bj^2$. On the other hand if α and β are consecutive zeros of u on I_j, then the right-hand inequality in Lemma 2 (which does not require $g\geqslant 0$) and the mean-value theorem for integrals yield the following for some $\eta\in I_j$:

$$\frac{\pi}{a}\leqslant(\beta-\alpha)\,f(\eta)^{1/2}\leqslant(\beta-\alpha)(j+1).$$

Hence $\beta-\alpha\geqslant\pi/(aj+a)$. If there are N zeros on I_j, the number of these intervals (α,β) is $N-1$, so

$$(N-1)\,\frac{\pi}{aj+a}\leqslant|I_j| \qquad \text{or} \quad N\leqslant 1+\left(1+\frac{1}{j}\right)\left(2+\frac{1}{j}\right)\frac{a}{\pi b}.$$

This is stronger than the assertion in Theorem 2.

If α and β are consecutive zeros of u, the equation $fu=-u''$ yields

$$0<-\frac{u'(\beta)}{u'(\alpha)}=\int_\alpha^\beta f(t)\,\frac{u(t)}{u'(\alpha)}\,dt-1.$$

Suppose now that $g\geqslant 0$. Estimating $u(t)/u'(\alpha)$ by Lemma 1 we get

$$-\frac{u'(\beta)}{u'(\alpha)}\leqslant\frac{1}{f(\alpha)^{1/4}}\int_\alpha^\beta f(t)^{3/4}\sin F(t)\,dt-1=I$$

where I is defined by this equation. Since $F'=f^{1/2}$, partial integration with

$$U=f^{1/4}, \qquad V=(\sin F)\,f^{1/2}\,dt$$

yields

$$f(\alpha)^{1/4}\,I\leqslant f(\beta)^{1/4}+\tfrac{1}{4}\int_\alpha^\beta f^{-3/4}f'\cos F\,dt.$$

Writing $f^{-3/4}f' = (f^{-5/4}f')\, f^{1/2}$ and integrating by parts again, we get

$$\int_\alpha^\beta f^{-3/4}f' \cos F\, dt = f(\beta)^{-5/4} f'(\beta) \sin F(\beta) + 4 \int_\alpha^\beta g f^{-1/4} \sin F\, dt.$$

With $r_n = |u'(\alpha_n)|\, f(\alpha_n)^{-1/4}$ the above estimates together yield

$$\frac{r_{n+1}}{r_n} \leqslant 1 + \delta_n + \eta_n, \qquad \delta_n = \frac{f'(\alpha_{n+1})}{4f(\alpha_{n+1})^{3/2}} \sin F_n(\alpha_{n+1})$$

$$\eta_n = \frac{1}{f(\alpha_n)^{1/4}} \int_{\alpha_n}^{\alpha_{n+1}} \frac{g(t)}{f(t)^{1/4}} \sin F_n(t)\, dt, \qquad F_n(t) = \int_{\alpha_n}^t \sqrt{f(s)}\, ds.$$

Convergence of $\prod r_{n+1}/r_n$ ensures that r_n is bounded, and we summarize as follows:

Lemma 3. *If $g \geqslant 0$ and $\sum \delta_n$ and $\sum \eta_n$ converge, then $|u'(\alpha_n)|\, f(\alpha_n)^{-1/4}$ admits a bound independent of n. Hence $u = O(f^{-1/4})$.*

The concluding statement follows from Lemma 1. We will use Lemma 1 to establish:

Theorem 3. *Suppose $-O(f')^2 \leqslant ff'' \leqslant (p+1)(f')^2$ where $0 < p < 1/4$. Suppose also $f' \geqslant bf$ and $g \geqslant 0$. Then $u = O(f^{-1/4})$.*

The hypothesis implies $(f^{-p})'' \geqslant 0$, hence $(f^{-p})'$ is increasing and $f' = O(f^{p+1})$. Together with the condition on f'' this gives

$$g = O\left(\frac{f'}{f}\right)^2 = O(f^{2p}), \qquad \frac{g}{\sqrt{f}} = O(f^{-\delta}) = O(e^{-b\,\delta t})$$

for some $\delta > 0$. Since f is increasing

$$\sum \eta_n \leqslant \sum \int_{\alpha_n}^{\alpha_{n+1}} \frac{g(t)}{f(t)^{1/2}}\, dt < \infty.$$

Theorem 3 now follows from Wintner's theorem. However an independent proof can be given by showing that $\sum \delta_n$ also converges.

The condition $g \geqslant 0$ gives $f' = O(f^{5/4})$. In Lemma 2 we assume for simplicity that $\alpha, \beta \in I_j$ and we take $a^2 - 1$ to be the maximum value of g/f on I_j. Note that $a^2 - 1 = O(1/j)$. Since $f' = O(f^{5/4}) = O(j^{5/2})$ on I_j, this gives

$$\delta_n \leqslant O(j^{5/2} j^{-3} j^{-1}) = O(j^{-3/2}),$$

where the factor j^{-1} comes from the sine. Theorem 2 now shows that $\sum \delta_n < \infty$, which completes the proof. The case $\alpha \in I_{j-1}$, $\beta \in I_j$ presents little additional difficulty.

The follwing example was needed in [2]. Let

$$\psi_1(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(n!)^2}, \qquad \psi_2(t) = \sum_{n=0}^{\infty} 2s_n(-1)^n \frac{t^n}{(n!)^2}$$

where $s_n = 1 + 1/2 + \cdots + 1/n$. Then the functions

$$\phi_1(t) = \psi_1(e^t), \qquad \phi_2(t) = \psi_2(e^t) - t\psi_1(e^t)$$

are linearly independent solutions of $u'' + e^t u = 0$. Theorem 3 gives $\phi_i = O(e^{-1/4})$. Hence $\psi_1(t) = O(t^{-1/4})$, $\psi_2(t) = O(t^{-1/4} \log t)$.

For $p > 0$ we denote by A_p the class of functions satisfying

$$f > 0, \qquad ff'' \leqslant (p+1)(f')^2, \qquad t \gg 1.$$

Since this class plays a prominent role in the foregoing theory, we conclude by giving some of its properties.

(i) $e^f \in A_p \Leftrightarrow f'' \leqslant p(f')^2$.

(ii) If f is a polynomial function with positive leading term, then $f \in A_p$, $e^f \in A_p$.

(iii) $f \in A_p \Leftrightarrow f^m \in A_{p/m}$ for $m > 0$.

(iv) $f \in A_p$, $\phi \in A_q$, $t\phi'(t) \geqslant \phi(t) \Rightarrow \phi(f) \in A_{p+q}$.

If E denotes the class of functions ϕ satusfying $\phi' \geqslant \delta\phi$ for $t \gg 1$ and some $\delta > 0$, we have also

(v) $f \in A_p, f \to \infty$, $\phi \in A_q \cap E$, $r > q \Rightarrow \phi(f) \in A_r$.

We prove only (iv) and (v). If $\phi'(t) > 0$ and $\phi \in A_q$, then

$$\phi(f)\, f'' \leqslant (r-q)\, \phi'(f)(f')^2 \Rightarrow \phi(f) \in A_r.$$

Replacing $\phi'(f)$ on the left by $\phi(f)/f$, we get (iv). For (v) we replace ϕ' by $\delta\phi$ and note that $f'' = o(f')^2$.

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